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ON RATE-DEPENDENT PLASTICITY—LINEAR VISCOELASTIC CONSTITUTIVE EQUATIONS

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Abstract—Based on the previous work by the authors in which generalized constitutive equations and evolution laws for rate-independent plasticity were developed from the theory of internal state variables in irreversible thermodynamics, the present paper establishes rate-dependent constitutive relations which have more generality than those proposed by other authors. As a particular case, generalized linear viscoelastic constitutive relations are derived under the assumption that the process is isothermal, which are found to be identical with those obtained from the general linear viscoelastic model analysis.

INTRODUCTION

Although the development of plasticity has lasted for more than a century, the interest in the establishment of plastic constitutive relations for different kinds of material has never decreased. In particular, the work of Duhem (1911) provided a theoretical foundation for the evolution of a new subject called continuum thermodynamics in which the concept of internal state variables is introduced to describe irreversible processes in continuum. The pioneering work to develop stress–strain relations based on this theory of internal state variables was done by Biot (1954) and Meixner (1954) for the case of infinitesimal deformation. Later, Coleman and Gurtin (1967), Onat (1966), Rice (1971), Hill and Rice (1973), and Valanis (1966) extended the approach to finite deformation.

Unlike the theory of the above authors, Deng *et al.* (1992) selected not only internal state variables but also inelastic strain as independent variables to characterize inelastic deformation. The Drucker postulates were generalized to overcome the limitation of the classical approach, which is unable to describe "real" hardening when kinematic hardening occurs. As a result, the rate-independent constitutive equations and evolution laws are derived.

In this paper, the constitutive equations and evolution laws for rate-dependent plasticity are established by introducing a free energy function and thermodynamic dissipation potential. Assuming that both the function and potential above are quadratic forms of their independent variables, the generalized linear viscoelastic constitutive equations are derived which are similar to but more general than those proposed by Biot, and are identical to those obtained from the general viscoelastic model analysis.

1. RATE-DEPENDENT CONSTITUTIVE EQUATIONS AND EVOLUTION LAWS

In continuum thermodynamics, there are two basic laws which play important roles in the development of many subjects related to thermodynamics. The two laws are the energy conservation law and the entropy increment inequality, and can be mathematically expressed as

$$\rho \dot{e} = \sigma_{ij}\dot{\varepsilon}_{ij} + r - q_{i,i} \tag{1}$$

and

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$$\rho \dot{s} - \frac{r}{T} + \left(\frac{q_i}{T}\right)_{,i} \geqslant 0 \tag{2}$$

respectively, where ρ is volume density, e is specific internal energy, r is volume density of the internal heat production, T is the absolute temperature, s is specific entropy, q_i (i ranges over x, y, z) is heat flux vector, and $\boldsymbol{\sigma}_{ij}$ and $\boldsymbol{\varepsilon}_{ij}$ (i, j range over x, y, z) are the stress and strain tensors for small deformation, respectively.

For convenience, we introduce the free energy function

$$\psi = e - Ts. \tag{3}$$

Substituting eqn (3) into eqns (1) and (2) yields

$$\boldsymbol{\sigma}_{ij}\dot{\boldsymbol{\varepsilon}}_{ij} - \rho(\dot{\psi} + s\dot{T}) - \rho T\dot{s} + r - q_{i,i} = 0 \tag{4}$$

and

$$\boldsymbol{\sigma}_{ij}\dot{\boldsymbol{\varepsilon}}_{ij} - \rho(\dot{\psi} + s\dot{T}) - q_i \frac{T_{,i}}{T} \geqslant 0 \tag{5}$$

where eqn (5) is called the Clausius–Duhem inequality.

Like the specific entropy s and the specific internal energy e, the free energy function ψ is taken as a state function in continuum thermodynamics which can be expressed by a set of basic state variables. The choice of the basic state variables is very important because they reflect the average macroscopic effects of the movement of a large number of particles. Nearly all other authors select ψ to be a function of the total strain tensor $\boldsymbol{\varepsilon}_{ij}$ or elastic strain tensor $\boldsymbol{\varepsilon}^{\mathrm{e}}_{ij}$, the temperature T and a suitable set of internal state variables. They indicate that the inelastic deformation in materials can be completely described by the chosen internal state variables. Indeed, this is true when kinematic hardening of materials does not occur or can be neglected. However, as shown by Deng *et al.* (1992), if the kinematic hardening is sufficiently large, we should choose, besides a set of internal state variables v_k ($k = 1, 2, \ldots, n$), the inelastic strain $\boldsymbol{\varepsilon}^{\mathrm{in}}_{ij}$ as an independent variable to characterize inelastic deformation. Thus, it is more reasonable to write ψ in the following form

$$\psi = \psi(\boldsymbol{\varepsilon}^{\mathrm{e}}_{ij}, \boldsymbol{\varepsilon}^{\mathrm{in}}_{ij}, v_k, T). \tag{6}$$

Most phenomenological experiments show that ψ can be expressed as the summation of a thermo-elastic term $\psi_1(\boldsymbol{\varepsilon}^{\mathrm{e}}_{ij}, T)$ and a thermo-inelastic term $\psi_2(\boldsymbol{\varepsilon}^{\mathrm{in}}_{ij}, v_k, T)$, (Lemaitre and Chaboche, 1990), i.e.

$$\psi = \psi_1(\boldsymbol{\varepsilon}^{\mathrm{e}}_{ij}, T) + \psi_2(\boldsymbol{\varepsilon}^{\mathrm{in}}_{ij}, v_k, T). \tag{7}$$

Equation (7) reflects the decoupling between reversible thermo-elastic and irreversible thermo-inelastic behaviour.

By substituting eqn (7) into eqns (4) and (5), and noting that $\boldsymbol{\varepsilon}^{\mathrm{e}}_{ij}$ and T are controllable variables, and that $\boldsymbol{\varepsilon}_{ij} = \boldsymbol{\varepsilon}^{\mathrm{e}}_{ij} + \boldsymbol{\varepsilon}^{\mathrm{in}}_{ij}$, we have

$$\boldsymbol{\sigma}_{ij} = \rho \frac{\partial \psi_1}{\partial \boldsymbol{\varepsilon}^{\mathrm{e}}_{ij}} \tag{8}$$

and

$$s = -\left(\frac{\partial\psi_1}{\partial T}+\frac{\partial\psi_2}{\partial T}\right) \tag{9}$$

while eqns (4) and (5) reduce to

$$\left(\sigma_{ij}-\rho\frac{\partial\psi_2}{\partial\varepsilon_{ij}^{\text{in}}}\right)\dot{\varepsilon}_{ij}^{\text{in}}-\rho\frac{\partial\psi_2}{\partial v_k}\dot{v}_k+r-\rho T\dot{s}-q_{i,i}=0 \tag{10}$$

and

$$\left(\sigma_{ij}-\rho\frac{\partial\psi_2}{\partial\varepsilon_{ij}^{\text{in}}}\right)\dot{\varepsilon}_{ij}^{\text{in}}-\rho\frac{\partial\psi_2}{\partial v_k}\dot{v}_k-q_i\frac{T_{,i}}{T}\geqslant 0 \tag{11}$$

respectively.

Equations (8) and (9) establish the thermo-elastic and entropy–temperature constitutive relations, respectively. Equation (10) is the heat transfer equation, while eqn (11) is called the dissipation inequality. It is obvious that the total number of derived equations, i.e. eqns (8)–(10) is less than the number of unknown variables. To solve a thermodynamic problem completely, complementary equations have to be sought.

Following the previous work of the present authors (1992), we define here that

$$\begin{cases} \sigma_{ij}^{\text{in}}=\sigma_{ij}-\rho\dfrac{\partial\psi_2}{\partial\varepsilon_{ij}^{\text{in}}} \\ W_k=-\rho\dfrac{\partial\psi_2}{\partial v_k} \qquad (k=1,2,\ldots,n) \\ G_i=-\dfrac{T_{,i}}{T} \end{cases} \tag{12a}$$

or

$$\begin{cases} \sigma_{ij}^{\text{in}}+\rho\dfrac{\partial\psi_2}{\partial\varepsilon_{ij}^{\text{in}}}=\sigma_{ij} \\ W_k+\rho\dfrac{\partial\psi_2}{\partial v_k}=0 \qquad (k=1,2,\ldots,n) \\ G_i+\dfrac{T_{,i}}{T}=0 \end{cases} \tag{12b}$$

where σ_{ij}^{in}, W_k and G_i are called thermodynamic forces associated with $\varepsilon_{ij}^{\text{in}}$, v_k and T respectively.

As many other authors have done, e.g. Lemaitre and Chaboche (1990), we also introduce a dissipation potential $\varphi(\sigma_{ij}^{\text{in}}, W_k, G_i)$ which is a function of thermodynamic forces. The evolution laws expressing the relations between flux variables and forces for rate-dependent plasticity can be given in the form

$$\begin{cases} \dot{\boldsymbol{\varepsilon}}_{ij}^{\text{in}} = \dfrac{\partial \varphi}{\partial \boldsymbol{\sigma}_{ij}^{\text{in}}} \\ \dot{v}_k = \dfrac{\partial \varphi}{\partial W_k} \qquad (k = 1, 2, \ldots, n). \\ q_i = \dfrac{\partial \varphi}{\partial G_i} \end{cases} \tag{13a}$$

Using the Legendre–Fenchel transform $\varphi^* = (\boldsymbol{\sigma}_{ij}^{\text{in}}\dot{\boldsymbol{\varepsilon}}_{ij}^{\text{in}} + W_k\dot{v}_k + G_i q_i) - \varphi$, we obtain a new potential $\varphi^*(\dot{\boldsymbol{\varepsilon}}_{ij}^{\text{in}}, \dot{v}_k, q_i)$, which produces an equivalent form of eqn (13a) as

$$\begin{cases} \boldsymbol{\sigma}_{ij}^{\text{in}} = \dfrac{\partial \varphi^*}{\partial \dot{\boldsymbol{\varepsilon}}_{ij}^{\text{in}}} \\ W_k = \dfrac{\partial \varphi^*}{\partial \dot{v}_k} \qquad (k = 1, 2, \ldots, n). \\ G_i = \dfrac{\partial \varphi^*}{\partial q_i} \end{cases} \tag{13b}$$

The evolution laws in eqn (13) provide complementary equations necessary for the complete solution of a thermodynamic problem under specified initial and boundary conditions. On the other hand, using eqn (12a), eqn (11) becomes

$$\boldsymbol{\sigma}_{ij}^{\text{in}}\dot{\boldsymbol{\varepsilon}}_{ij}^{\text{in}} + W_k\dot{v}_k + G_i q_i \geqslant 0 \tag{14}$$

which shows that the total dissipation work done by the associated forces on their corresponding thermodynamic variables will never be negative.

It is apparent by substituting eqn (13a) or (13b) into eqn (14) that the inequality in eqn (14) can be automatically satisfied if φ or φ^* is a non-negative, convex function which contains the origin $[\varphi(0,0,0) = 0, \varphi^*(0,0,0) = 0]$.

Also it is obvious that when inelastic strain $\varepsilon_{ij}^{\text{in}}$ does not explicitly appear in the free energy function ψ, $\sigma_{ij}^{\text{in}} = \sigma_{ij}$. Thus all the results in this section reduce to the ones by other authors. Because the constitutive equations given in eqns (8)–(10) and evolution laws in eqn (13) contain a set of internal state variables, we need to eliminate them by using the above simultaneous equations to obtain the stress–strain constitutive relations. In the following section, we will discuss the derivation of stress–strain constitutive equations for viscoelastic materials under the assumption that the process is isothermal and the deformation is small.

2. GENERALIZED LINEAR VISCOELASTIC CONSTITUTIVE EQUATIONS

In engineering, there are a lot of materials such as plastics, rubber, resin, glass and metal which, when loaded under high temperature, exhibit a particular behaviour between elasticity and viscosity called viscoelasticity. A significant amount of work has been done in the past on the viscoelastic properties in both model analysis and experimental study. A systematic discussion on the subject can be found in references by Ward (1983), Ferry (1980), Christensen (1982), and Flügge (1975).

Biot (1954) developed linear viscoelastic stress–strain constitutive relations based on the theory of internal state variables in thermodynamics. In his paper, however, only the total strain was taken as an independent state variable. In fact, as discussed in the previous section, it is more reasonable to consider both elastic and inelastic strains as two independent state variables.

Assuming the loading process is slow enough to be regarded as isothermal, then T keeps constant, $G_i = 0$, and $q_i = 0$. This means that no heat dissipation takes place. From eqns (12b) and (13b) we have

$$\begin{cases} \dfrac{\partial \varphi^*}{\partial \dot{\boldsymbol{\varepsilon}}_{ij}^{\text{in}}} + \rho \dfrac{\partial \psi_2}{\partial \boldsymbol{\varepsilon}_{ij}^{\text{in}}} = \boldsymbol{\sigma}_{ij} \\ \dfrac{\partial \varphi^*}{\partial \dot{v}_k} + \rho \dfrac{\partial \psi_2}{\partial v_k} = 0 \end{cases} \quad (k = 1, 2, \ldots, n) \tag{15}$$

and eqn (14) becomes

$$\boldsymbol{\sigma}_{ij}^{\text{in}} \dot{\boldsymbol{\varepsilon}}_{ij}^{\text{in}} + W_k \dot{v}_k \geqslant 0. \tag{16}$$

It is well known from Hooke's law that for linear elastic materials, the relation between stress and strain is linear. That is, strain energy density is a quadratic form of strain. Similarly, for many viscoelastic materials, all the potentials $\psi_1(\boldsymbol{\varepsilon}_{ij}^{\text{e}})$, $\psi_2(\boldsymbol{\varepsilon}_{ij}^{\text{in}}, v_k)$ and $\varphi^*(\dot{\boldsymbol{\varepsilon}}_{ij}^{\text{in}}, \dot{v}_k)$ are also quadratic forms of their corresponding independent variables from phenomenological experiments.

For convenience, noting the symmetry of stress and strain tensors, and using Onsager's symmetry relations, we can write the potentials as follows

$$\psi_1 = \frac{1}{2\rho} \sum_{i,j=1}^{6} E_{ij} \boldsymbol{\varepsilon}_i^{\text{e}} \boldsymbol{\varepsilon}_j^{\text{e}} \tag{17}$$

$$\psi_2 = \frac{1}{2\rho} \sum_{i,j=1}^{m} A_{ij} \boldsymbol{\varepsilon}_i^{\text{in}} \boldsymbol{\varepsilon}_j^{\text{in}} \tag{18}$$

$$\varphi^* = \frac{1}{2} \sum_{i,j=1}^{m} B_{ij} \dot{\boldsymbol{\varepsilon}}_i^{\text{in}} \dot{\boldsymbol{\varepsilon}}_j^{\text{in}}. \tag{19}$$

Thus eqns (8), (15) and (16) become

$$\sum_{j=1}^{m} E_{ij}^* \boldsymbol{\varepsilon}_j^{\text{e}} = \boldsymbol{\sigma}_i \quad (i = 1, 2, \ldots, m) \tag{20}$$

$$\sum_{j=1}^{m} A_{ij} \boldsymbol{\varepsilon}_j^{\text{in}} + \sum_{j=1}^{m} B_{ij} \dot{\boldsymbol{\varepsilon}}_j^{\text{in}} = \boldsymbol{\sigma}_i \quad (i = 1, 2, \ldots, m) \tag{21}$$

$$\sum_{i,j=1}^{m} B_{ij} \dot{\boldsymbol{\varepsilon}}_i^{\text{in}} \dot{\boldsymbol{\varepsilon}}_j^{\text{in}} \geqslant 0 \tag{22}$$

respectively.

The matrix forms of eqns (20) and (21) are

$$[\mathbf{E}^*]\{\boldsymbol{\varepsilon}^{\text{e}}\} = \{\boldsymbol{\sigma}\} \tag{23}$$

and

$$[\mathbf{A}]\{\boldsymbol{\varepsilon}^{\text{in}}\} + [\mathbf{B}]\{\dot{\boldsymbol{\varepsilon}}^{\text{in}}\} = \{\boldsymbol{\sigma}\} \tag{24}$$

where

$$\begin{aligned}\{\boldsymbol{\varepsilon}^{\mathrm{e}}\} &= (\varepsilon_1^{\mathrm{e}} \quad \varepsilon_2^{\mathrm{e}} \quad \varepsilon_3^{\mathrm{e}} \quad \varepsilon_4^{\mathrm{e}} \quad \varepsilon_5^{\mathrm{e}} \quad \varepsilon_6^{\mathrm{e}} \quad \varepsilon_7^{\mathrm{e}} \quad \ldots \quad \varepsilon_m^{\mathrm{e}})^{\mathrm{T}} \\ &= (\varepsilon_{xx}^{\mathrm{e}} \quad \varepsilon_{yy}^{\mathrm{e}} \quad \varepsilon_{zz}^{\mathrm{e}} \quad 2\varepsilon_{yz}^{\mathrm{e}} \quad 2\varepsilon_{zx}^{\mathrm{e}} \quad 2\varepsilon_{xy}^{\mathrm{e}} \quad 0 \quad \ldots \quad 0)^{\mathrm{T}}\end{aligned} \tag{25}$$

$$\begin{aligned}\{\boldsymbol{\varepsilon}^{\mathrm{in}}\} &= (\varepsilon_1^{\mathrm{in}} \quad \varepsilon_2^{\mathrm{in}} \quad \varepsilon_3^{\mathrm{in}} \quad \varepsilon_4^{\mathrm{in}} \quad \varepsilon_5^{\mathrm{in}} \quad \varepsilon_6^{\mathrm{in}} \quad \varepsilon_7^{\mathrm{in}} \quad \ldots \quad \varepsilon_m^{\mathrm{in}})^{\mathrm{T}} \\ &= (\varepsilon_{xx}^{\mathrm{in}} \quad \varepsilon_{yy}^{\mathrm{in}} \quad \varepsilon_{zz}^{\mathrm{in}} \quad 2\varepsilon_{yz}^{\mathrm{in}} \quad 2\varepsilon_{zx}^{\mathrm{in}} \quad 2\varepsilon_{xy}^{\mathrm{in}} \quad v_1 \quad \ldots \quad v_n)^{\mathrm{T}}\end{aligned} \tag{26}$$

$$\begin{aligned}\{\boldsymbol{\sigma}\} &= (\sigma_1 \quad \sigma_2 \quad \sigma_3 \quad \sigma_4 \quad \sigma_5 \quad \sigma_6 \quad \sigma_7 \quad \ldots \quad \sigma_m)^{\mathrm{T}} \\ &= (\sigma_{xx} \quad \sigma_{yy} \quad \sigma_{zz} \quad \sigma_{zx} \quad \sigma_{xy}0 \quad \ldots \quad 0)^{\mathrm{T}}\end{aligned} \tag{27}$$

and **[E*]**, **[A]**, **[B]** are all symmetrical square matrices of order $m \times m$ $(m = 6+n)$:

$$[\mathbf{E}^*]^{\mathrm{T}} = [\mathbf{E}^*] = \begin{bmatrix} [\mathbf{E}] & [0] \\ [0] & [\mathbf{I}] \end{bmatrix} \tag{28}$$

$$[\mathbf{A}]^{\mathrm{T}} = [\mathbf{A}] = \begin{bmatrix} A_{11} & A_{12} & \ldots & A_{1m} \\ A_{21} & A_{22} & \ldots & A_{2m} \\ \ldots & \ldots & \ldots & \ldots \\ A_{m1} & A_{m2} & \ldots & A_{mm} \end{bmatrix} \tag{29}$$

$$[\mathbf{B}]^{\mathrm{T}} = [\mathbf{B}] = \begin{bmatrix} B_{11} & B_{12} & \ldots & B_{1m} \\ B_{21} & B_{22} & \ldots & B_{2m} \\ \ldots & \ldots & \ldots & \ldots \\ B_{m1} & B_{m2} & \ldots & B_{mm} \end{bmatrix} \tag{30}$$

while **[I]** is an identity matrix of order $n \times n$, and **[E]** is the elastic matrix of order 6×6

$$[\mathbf{E}] = \begin{bmatrix} E_{11} & E_{12} & \ldots & E_{16} \\ E_{21} & E_{22} & \ldots & E_{26} \\ \ldots & \ldots & \ldots & \ldots \\ E_{61} & E_{62} & \ldots & E_{66} \end{bmatrix}. \tag{31}$$

Note that the equality sign in eqn (22) holds only for purely elastic deformation. For viscoelastic materials being discussed, φ^* is always positive, i.e. φ^* is a quadratic form of positive definite, so its coefficient matrix **[B]** is real, symmetrical and positive.

Now let us establish constitutive relations between stress $\{\boldsymbol{\sigma}\}$ and total strain $\{\boldsymbol{\varepsilon}\}$ by using eqns (23) and (24). As **[E*]** is a symmetrical and non-singular matrix, its inverse **[S]** must exist. In the light of the rule for the inverse of a symmetrical matrix by partitioning, Kraus (1987), we know that **[S]** is also a symmetrical square matrix of order $m \times m$

$$[\mathbf{S}]^{\mathrm{T}} = [\mathbf{S}] = [\mathbf{E}^*]^{-1} = \begin{bmatrix} [\mathbf{E}]^{-1} & [0] \\ [0] & [\mathbf{I}] \end{bmatrix}. \tag{32}$$

Thus eqn (23) may be written as

$$\{\boldsymbol{\varepsilon}^{\mathrm{e}}\} = [\mathbf{S}]\{\boldsymbol{\sigma}\}. \tag{33}$$

Noting that $\{\boldsymbol{\varepsilon}^{\mathrm{in}}\} = \{\boldsymbol{\varepsilon}\} - \{\boldsymbol{\varepsilon}^{\mathrm{e}}\}$, we can express eqn (24) as follows

$$[\mathbf{A}](\{\varepsilon\}-\{\varepsilon^{e}\})+[\mathbf{B}](\{\dot{\varepsilon}\}-\{\dot{\varepsilon}^{e}\}) = \{\sigma\}. \tag{34}$$

The substitution of eqn (33) into eqn (34) produces

$$[\mathbf{A}]\{\varepsilon\}+[\mathbf{B}]\{\dot{\varepsilon}\} = [\mathbf{C}]\{\sigma\}+[\mathbf{D}]\{\dot{\sigma}\} \tag{35}$$

where

$$[\mathbf{C}] = [\mathbf{A}][\mathbf{S}]+[\mathbf{I}^{*}], \quad [\mathbf{D}] = [\mathbf{B}][\mathbf{S}] \tag{36}$$

with $[\mathbf{I}^{*}]$ being an identity matrix of $m \times m$.

Equation (35) is a matrix differential equation relating stress with total strain. It is obvious that if we neglect the influence of $\{\varepsilon^{e}\}$, thus $\{\varepsilon^{in}\} = \{\varepsilon\}$, and eqn (35) reduces to

$$[\mathbf{A}]\{\varepsilon\}+[\mathbf{B}]\{\dot{\varepsilon}\} = \{\sigma\}. \tag{37}$$

Comparing eqn (35) with eqn (37) we see that the left hand sides of the two equations are identical but the right hand sides are different. Equation (35) includes not only the term of stress but its derivative as well, which, although being a matrix equation, is similar to a standard linear solid (or three parameter solid) model in form while eqn (37) includes only the term of stress, which is similar to a Kelvin–Voigt model. In the following, we will show that the solution of eqn (35) gives the constitutive relations of the general Kelvin–Voigt solid which has been used for describing linear viscoelastic behaviour in polymers and metals at high temperature.

2.1. *Solution of homogeneous equations*

Let us first consider the solution of homogeneous equations [eqn (35)]

$$[\mathbf{A}]\{\varepsilon\}+[\mathbf{B}]\{\dot{\varepsilon}\} = \{0\}. \tag{38}$$

The general form for the solution of the above matrix equation is

$$\{\varepsilon\} = \{\phi\}\,\mathrm{e}^{-\lambda t} \tag{39}$$

where

$$\{\phi\} = (\phi_1 \quad \phi_2 \quad \ldots \quad \phi_m)^{\mathrm{T}}. \tag{40}$$

Substituting eqn (39) into eqn (38) gives

$$([\mathbf{A}]-\lambda[\mathbf{B}])\,\{\phi\} = \{\mathbf{0}\}. \tag{41}$$

The characteristic equation corresponding to eqn (41) is

$$\det|A_{ij}-\lambda B_{ij}| = 0. \tag{42}$$

As $[\mathbf{B}]$ is real, symmetrical and positive, it can be partitioned as

$$[\mathbf{B}] = [\mathbf{L}][\mathbf{L}]^{\mathrm{T}} \tag{43}$$

where $[\mathbf{L}]$ is a non-singular, lower triangular matrix

$$[\mathbf{L}] = \begin{bmatrix} l_{11} & & & \\ l_{21} & l_{22} & & \\ \cdot & & \cdot & \\ \cdot & & & \cdot \\ \cdot & & & & \cdot \\ l_{m1} & l_{m2} & \cdots & & l_{mm} \end{bmatrix}. \tag{44}$$

Premultiplying eqn (41) by $[\mathbf{L}]^{-1}$ and defining $\{\tilde{\boldsymbol{\phi}}\} = [\mathbf{L}]^{\mathrm{T}}\{\boldsymbol{\phi}\}$, we have

$$([\tilde{\mathbf{A}}]-\lambda[\mathbf{I}])\{\tilde{\boldsymbol{\phi}}\} = \{\mathbf{0}\}. \tag{45}$$

As $[\tilde{\mathbf{A}}] = [\mathbf{L}]^{-1}[\mathbf{A}][\mathbf{L}]^{-\mathrm{T}}$ is symmetrical, all the eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_m$ are real, and there are m eigenvectors $\{\boldsymbol{\phi}^{(1)}\}, \{\boldsymbol{\phi}^{(2)}\}, \ldots, \{\boldsymbol{\phi}^{(m)}\}$ corresponding to the m eigenvalues. Any two of the m eigenvectors are linear independent and orthogonal with respect to matrix $[\mathbf{B}]$, i.e.

$$\{\boldsymbol{\phi}^{(i)}\}^{\mathrm{T}}[\mathbf{B}]\{\boldsymbol{\phi}^{(j)}\} = \delta_{ij} = \begin{cases} 0 & i \neq j \\ 1 & i = j \end{cases} \quad (i,j = 1,2,\ldots,m). \tag{46}$$

So from eqn (41)

$$\begin{aligned} \{\boldsymbol{\phi}^{(i)}\}^{\mathrm{T}}[\mathbf{A}]\{\boldsymbol{\phi}^{(j)}\} &= \lambda_j\{\boldsymbol{\phi}^{(i)}\}^{\mathrm{T}}[\mathbf{B}]\{\boldsymbol{\phi}^{(j)}\} = \lambda_j\delta_{ij} \\ &= \begin{cases} 0 & i \neq j \\ \lambda_j & i = j \end{cases} \quad (i,j = 1,2,\ldots,m). \end{aligned} \tag{47}$$

The general solution of the homogeneous equations [eqn (38)] is given in the following form

$$\{\boldsymbol{\varepsilon}\} = \sum_{j=1}^{m} \{\boldsymbol{\phi}^{(j)}\}\,\mathrm{e}^{-\lambda_j t} \tag{48}$$

where $\tau_j = 1/\lambda_j$ is called the retardation time constant, while $\{\boldsymbol{\phi}^{(j)}\}$ is sometimes called the creep mode corresponding to $\lambda_j (j = 1,2,\ldots,m)$.

If we let

$$[\boldsymbol{\phi}]_{\mathrm{M}} = [\{\boldsymbol{\phi}^{(1)}\}\{\boldsymbol{\phi}^{(2)}\}\ldots\{\boldsymbol{\phi}^{(m)}\}] \tag{49}$$

it follows from eqns (46) and (47) that

$$[\boldsymbol{\phi}]_{\mathrm{M}}^{\mathrm{T}}[\mathbf{B}][\boldsymbol{\phi}]_{\mathrm{M}} = [\mathbf{I}] \tag{50}$$

$$[\boldsymbol{\phi}]_{\mathrm{M}}^{\mathrm{T}}[\mathbf{A}][\boldsymbol{\phi}]_{\mathrm{M}} = [\boldsymbol{\lambda}] \tag{51}$$

where $[\boldsymbol{\lambda}]$ denotes a diagonal matrix of $m \times m$ with λ_i being the diagonal elements

$$[\boldsymbol{\lambda}] = \begin{bmatrix} \lambda_1 & & & \\ & \lambda_2 & & \\ & & \ddots & \\ & & & \lambda_m \end{bmatrix}. \tag{52}$$

2.2. *Solution of nonhomogeneous equations*

To obtain the general solution of the nonhomogeneous equations [eqn (35)], let us introduce so called normalized coordinates

$$\{\boldsymbol{\eta}\} = (\eta_1 \quad \eta_2 \quad \dots \quad \eta_m)^{\mathrm{T}}. \tag{53}$$

The strain components can be expressed by $\{\boldsymbol{\eta}\}$ as

$$\{\boldsymbol{\varepsilon}\} = [\boldsymbol{\phi}]_{\mathrm{M}}\{\boldsymbol{\eta}\}. \tag{54}$$

Inserting eqn (54) into eqn (35) allows one to write

$$[\mathbf{A}][\boldsymbol{\phi}]_{\mathrm{M}}\{\boldsymbol{\eta}\} + [\mathbf{B}][\boldsymbol{\phi}]_{\mathrm{M}}\{\dot{\boldsymbol{\eta}}\} = [\mathbf{C}]\{\boldsymbol{\sigma}\} + [\mathbf{D}]\{\dot{\boldsymbol{\sigma}}\}. \tag{55}$$

Premultiplying eqn (55) by $[\boldsymbol{\phi}]_{\mathrm{M}}^{\mathrm{T}}$ and using eqns (50) and (51) gives

$$[\boldsymbol{\lambda}]\{\boldsymbol{\eta}\} + [\mathbf{I}]\{\dot{\boldsymbol{\eta}}\} = [\mathbf{F}]\{\boldsymbol{\sigma}\} + [\mathbf{H}]\{\dot{\boldsymbol{\sigma}}\} \tag{56}$$

where

$$[\mathbf{F}] = [\boldsymbol{\phi}]_{\mathrm{M}}^{\mathrm{T}}[\mathbf{C}], \quad [\mathbf{H}] = [\boldsymbol{\phi}]_{\mathrm{M}}^{\mathrm{T}}[\mathbf{D}]. \tag{57}$$

In view of the decoupling between the displacement and velocity terms of the normalized coordinates in eqn (56), we have

$$\lambda_i\eta_i + \dot{\eta}_i = \sum_{j=1}^{m} \mathrm{F}_{ij}\sigma_j + \sum_{j=1}^{m} \mathrm{H}_{ij}\dot{\sigma}_j \quad (i = 1, 2, \dots, m). \tag{58}$$

Taking the Laplace transform for eqn (58) and utilizing the initial condition $\{\eta(t=0)\} = \{\boldsymbol{\sigma}(t=0)\} = \{0\}$, we have

$$\bar{\eta}_i = \sum_{j=1}^{m} H_{ij}\bar{\sigma}_j + \sum_{j=1}^{m} \frac{(F_{ij} - H_{ij}\lambda_i)}{\lambda_i + s}\bar{\sigma}_j \quad (i = 1, 2, \dots, m). \tag{59}$$

The inverse transform of eqn (59) gives

$$\eta_i(t) = \sum_{j=1}^{m} H_{ij}\sigma_j(t) + \sum_{j=1}^{m} (F_{ij} - H_{ij}\lambda_i)\int_0^t \sigma_j(t-\xi)\,\mathrm{e}^{-\lambda_i\xi}\,\mathrm{d}\xi \quad (i = 1, 2, \dots, m). \tag{60}$$

Substituting eqn (60) into eqn (54), and noting that only the first six components in $\sigma_j (j = 1, 2, \dots, m)$ do not vanish, we obtain

$$\begin{aligned} \{\boldsymbol{\varepsilon}\} &= \sum_{i=1}^{m} \{\boldsymbol{\phi}^{(i)}\}\eta_i \\ &= \sum_{i=1}^{m} \left\{ \{\boldsymbol{\phi}^{(i)}\} \left[\sum_{j=1}^{6} H_{ij}\sigma_j(t) + \sum_{j=1}^{6} (F_{ij} - H_{ij}\lambda_i) \int_0^t \sigma_j(t-\xi)\,\mathrm{e}^{-\lambda_i\xi}\,\mathrm{d}\xi \right] \right\}. \end{aligned} \tag{61}$$

Hence, eqn (61) offers the complete solution of differential equations of eqn (35), which has a similar form to the solution of eqn (37) given by Biot but is more general because the solution of eqn (37) can be deduced from eqn (61) by letting $[\mathbf{F}] = [\boldsymbol{\phi}]_{\mathrm{M}}^{\mathrm{T}}$ and $[\mathbf{H}] = [\mathbf{0}]$.

DISCUSSION

As a new model, it is usually necessary to compare its predicted results with those from the experiments. Fortunately, it is easily found that eqn (61) is equal to the constitutive relation of a generalized Kelvin–Voigt solid. This model is formed by connecting a spring and m Kelvin–Voigt elements in series (Fig. 1), and describes the mechanical behaviour of general viscoelastic materials. The validity of this model has been verified by many experimental results. See, for example, Shames and Cozzarelli (1992), and Aklonis and Macknight (1983). This indicates that the predicted results from eqn (61) would be in good agreement with the experimental results.

The stress–strain relation for uniaxial stress can be easily derived from eqn (61). Consider the following uniaxial stress state in which

$$\sigma_1 = \sigma_1(t), \quad \sigma_2 = \sigma_3 = \cdots = \sigma_6 = 0. \tag{62}$$

A substitution of the above equation into eqn (61) gives

$$\varepsilon_1 = c\sigma_1(t) + \sum_{i=1}^{m} d_i \int_0^t \sigma_1(t-\xi)\,\mathrm{e}^{-\lambda_i \xi}\,\mathrm{d}\xi \tag{63}$$

where

$$c = \sum_{i=1}^{m} \phi_1^{(i)} H_{i1}, \quad d_i = \phi_1^{(i)}(F_{i1} - H_{i1}\lambda_i). \tag{64}$$

For $\sigma_1(t) = \sigma_0 u(t)$ where σ_0 is a constant and $u(t)$ is a unit step function, eqn (63) becomes

$$\varepsilon_1 = \left[c + \sum_{i=1}^{m} \frac{d_i}{\lambda_i}(1 - \mathrm{e}^{-\lambda_i t})\right]\sigma_0 u(t). \tag{65}$$

The creep compliance function $J(t)$ is derived as

$$J(t) = \frac{\varepsilon_1}{\sigma_0} = \left[c + \sum_{i=1}^{m} \frac{d_i}{\lambda_i}(1 - \mathrm{e}^{-\lambda_i t})\right]u(t). \tag{66}$$

The same expression for $J(t)$ can also be obtained by using the Boltzmann superposition principle for the generalized Kelvin–Voigt solid model shown in Fig. 1.

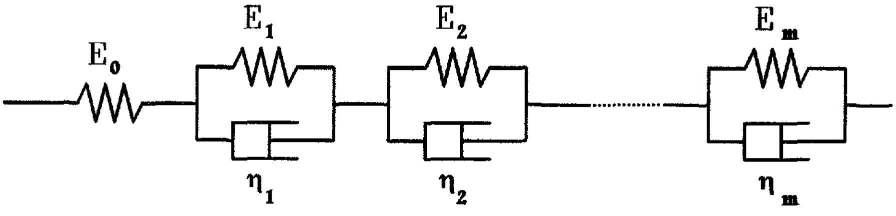

Fig. 1. Generalized Kelvin–Voigt solid.

CONCLUSION

Generalized rate-dependent constitutive equations are established based on the theory of internal state variables in continuum thermodynamics by introducing a dissipation potential. Both inelastic strain and a set of internal variables are used to characterize irreversible deformation processes. Viscoelastic constitutive relations are developed as a particular case when the free energy function and dissipation potential are both taken as quadratic forms of their independent variables. The constructed relations are the same as those developed from the general linear viscoelastic model analysis.

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